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Superconductivity in the fractional-statistics gas

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Abstract. We show that a charged hard-core boson with fractional statistics might have anisotropic superconductivity. The vector potential of this magnetic field is singular and depends on the electric charge density. Mapping the problem to a vortex lattice we find that for the semion case the charge neutrality condition implies that the electric charge can have two opposite vortices $\pm 1/2$. As a result the two vortices bind forming a boson of charge $2e$. The critical temperature for binding is the Kosterlitz–Thouless one. Below this critical temperature an effective model for anisotropic superconductivity is derived.

The purpose of this paper is to investigate the effects of a statistical gauge field on a bose system. We show here that for the two-dimensional case the statistical field A that satisfies the condition [1, 2] $\nabla \times A = (1/\Theta)\rho(x)$ ($\rho(x) = 0$, 1 is the electric charge density) is changed to the statistical field $\boldsymbol{a}$ such that $\nabla \times \boldsymbol{a} = 2\pi[q(r) + (1/2\pi\Theta)\rho(x)]$ where $q(r) = 1, \pm 1, \pm 2 \ldots$ is the vorticity. When $\Theta = 1/\pi$ each electric charge carries half a vortex. Owing to the charge neutrality for each intrinsic vortex $q(r) = 0, \pm 1 \ldots$ one needs to electric charges (each one with half a vortex). It is this new condition which gives rise to binding (for $\Theta = 1/\pi$) of opposite vorticity with the same electric charge. As a result we find binding with the charge $2e$. The critical temperature for binding is identical to the Kosterlitz–Thouless transition.

We consider a hard-core bose gas in two dimensions in the presence of a statistical gauge field $A(x)$:

$$H = t \sum_{x,\hat{e}=1,2} b^+(x)\, \mathrm{e}^{\mathrm{i}A_{\hat{e}}(x)} b(x+\hat{e}) + \mathrm{HC} + \mu_0 \sum_x b^+(x)b(x) + V\sum_x (x) b^+(x)b(x)b(x) \tag{1}$$

where $\mu_0 = \mu_0(\delta)$ is the chemical potential for the bosons with concentration δ, and V is a hard-core repulsive interaction, which imposes the condition $\rho(x) = b^+(x)b(x) = 0, 1$. The statistical gauge potential is given by the constraint condition [1]

$$\rho(x) = \Theta\nabla \times A = \Theta B(r) \tag{2}$$

where $B(r)$ is the statistical magnetic field defined on the dual lattice, and Θ is the strength parameter of the Chern–Simons term [1]. When $\Theta = 1/\pi$ one finds that the interchange of two particles multiples the wavefunction by a phase $\exp(\mathrm{i}\pi/2)$. This case corresponds to the semion case (half-fermion, one has to interchange twice the particles).

When $\Theta = 1/2\pi$ the wavefunction is multiplied by $\exp(\mathrm{i}\pi)$. This case corresponds to a fermion.

In order to investigate equation (1) we use the coherent-state representation [3] of the Hamiltonian given in equation (1) and introduce

$$b(x) = \tilde{b}(x)\exp[\mathrm{i}\Theta_v(x)] = r(x)\exp[\mathrm{i}f(x)]. \tag{3}$$

The hard-core boson can be analysed with the use of the spin-coherent representation for pin 1/2. In this representation $r(x)$ is the amplitude and $f(x)$ is the phase.

Using the spin-1/2 representation [4] we have $r(x) = \frac{1}{2}\sin\alpha(x)$ and $\rho(x) \equiv b^+(x)b(x) \equiv \tilde{b}^+(x)(\tilde{b}(x) = \frac{1}{2} - S_z(x)$ with $S_z(x) \equiv \frac{1}{2}\cos\alpha(x)$. Here $r(x)$ is given in terms of the density

$$r(x) = \tfrac{1}{2}[1-\cos^2\alpha(x)]^{1/2} = \tfrac{1}{2}\{1-[1-2\rho(x)]^2\}^{1/2} = \{\rho(x)[1-\rho(x)]\}^{1/2}.$$

We separate the vorticity $\Theta_v(x)$ from the normal part $\psi(x)$:

$$f(x) \equiv \psi(x) + \Theta_v(x). \tag{4}$$

The functions $\psi(x)$ and $\Theta_v(x)$ satisfy the conditions

$$V^{(v)}(x) = \nabla\Theta_v(x) \qquad V^{(n)}(x) = \nabla\psi(x) \tag{5}$$

$$\nabla \times V^{(v)}(x) \neq 0 \qquad \nabla\cdot V^{(v)}(x) = 0 \qquad \nabla\times V_{(n)}(x) = 0. \tag{6}$$

Here $\Theta_v(x)$ represents the vorticity. Basically we assume that $b(x)$ vanishes at discrete points. Around such points we have

$$b(x) \sim \tilde{b}(x)\,\mathrm{e}^{\mathrm{i}\theta_v(x)} \sim \tilde{b}(x)(x+\mathrm{i}y)/(x^2+y^2)^{1/2} \qquad (|\nabla\Theta_v| \simeq 1/r).$$

The function $\tilde{b}(x)$ is smooth and the vanishing of $b(x)$ is absorbed into the term $\exp[\mathrm{i}\Theta_v(x)]$. The vector field $V^{(v)} = \nabla\Theta_v(x)$ can be expressed with the aid of a scalar function $\Phi^{(V)}(r)$

$$V_\mu^{(v)}(x) = \Delta_\mu\theta_v(x) = \varepsilon_{\mu\nu}\Delta_\nu\Phi^{(V)}(r) \tag{7}$$

where we have used the definition $\Delta_\mu f(x) = f(x+\mu) - f(x)$ and $\varepsilon_{11} = \varepsilon_{22} = 0$, $-\varepsilon_{12} = \varepsilon_{21} = -1$.

The vector field $V^{(v)}(x)$ satisfies

$$\nabla\times V^{(v)}(x) = \varepsilon_{\mu\nu}\Delta_\mu V_\nu^{(v)}(x) = \varepsilon_{\mu\nu}\Delta_\mu\varepsilon_{\mu\nu'}\Delta_{\mu'}\Phi^{(V)}(r) = \Delta^2\Phi^{(V)}(r) = 2\pi q(r)$$
$$q(r) = 0, \pm 1, \ldots \tag{8}$$

where $q(r)$ is the vorticity charge.

The presence of the statistical gauge field $A(x)$ is included by the appearance of a function $\exp[\mathrm{i}\varphi(x)]$. The phase $\varphi(x)$ is related to $A(x)$ by

$$A\mu(x) = \Delta_\mu\varphi(x) = \varphi(x+\mu) - \varphi(x). \tag{9}$$

The kinetic term can be expressed as

$$b^+(x)\exp[\mathrm{i}A_\mu(x)]b(x+\mu) = \tilde{b}^+(x)\exp[-\mathrm{i}\Theta_v(x) - \varphi(x)]$$
$$\times\exp[+\mathrm{i}\Theta_v(x+\mu) + \mathrm{i}\varphi(x+\mu)]\tilde{b}(x).$$

The vector field $A_\mu(x)$ can be represented with the aid of the scalar function $\Phi^{(A)}(r)$

in the same way that $V_\mu(x)$ (the vortex field) has been related to the scalar function $\Phi^{(V)}(r)$. Performing a similar analysis as in equations (7) and (8) we find

$$\begin{aligned} A_\mu(x) &= \Delta_\mu \varphi(x) = \varepsilon_{\mu\nu}\Delta_\nu \Phi^{(A)}(r) \\ \nabla \times A &\equiv \varepsilon_{\mu\nu}\Delta_\mu A_\nu \equiv \varepsilon_{\mu\nu}\Delta_\mu \varepsilon_{\mu\nu'}\Delta_{\mu'}\Phi^{(A)}(r) \equiv \Delta^2\Phi^{(A)}(r) = B(r). \end{aligned} \tag{10}$$

Combining the results given in equations (7), (8) and (10) we find

$$\begin{aligned} \nabla \times (A + \nabla\Theta_v) &\equiv \nabla \times \boldsymbol{a} = \Delta^2(\Phi^{(A)}(r) + \Phi^{(V)}(r)) = \Delta^2\Phi(r) \\ &= 2\pi[q(r) + (1/2\pi)B(r)] = 2\pi Q(r) \end{aligned} \tag{11}$$

where we have defined

$$A(x) + \nabla\Theta_v(x) \equiv \boldsymbol{a}(x) \qquad \Phi(r) \equiv \Phi^{(A)}(r) + \Phi^{(V)}(r) \tag{12}$$

with the property

$$\nabla \times \boldsymbol{a}(x) = 2\pi Q(r) = 2\pi[q(r) + (1/2\pi\Theta)\tilde{b}^+(x)\tilde{b}(x)] \tag{13}$$

$$\nabla \cdot \boldsymbol{a}(x) = 0. \tag{14}$$

$Q(r)$ represents the total vorticity composed from $q(r) = 0, \pm 1, \ldots$ (the intrinsic vorticity) and the external vorticity $B(r)/2\pi$ induced by the electric charge $(1/\Theta)\rho(x)$. Equation (13) shows that the statistical field $A(x)$ is replaced by the vector potential $\boldsymbol{a}(x)$. The Hamiltonian given in equation (1) takes the form

$$H = t\sum_{x,\mu} \tilde{b}(x)\,\mathrm{e}^{\mathrm{i}a_\mu(x)}\tilde{b}(x+\mu) + \mathrm{HC} + \mu_0\tilde{b}^+(x)\tilde{b}(x) + V\Sigma\tilde{b}^+(x)\tilde{b}^+(x)\tilde{b}(x)\tilde{b}(x). \tag{15}$$

In equation (15) we perform the continuum approximation

$$\begin{aligned} \sum_{x,\mu} \tilde{b}(x)\,\mathrm{e}^{\mathrm{i}a_\mu(x)}\tilde{b}(x+\mu) + \mathrm{HC} &\simeq \sum_{x,\mu} \tilde{b}^+(x)\tilde{b}(x+\mu) + \mathrm{HC} - \sum_x \tilde{b}^+(x)\tilde{b}(x)a_\mu^2(x) \\ &\quad + \mathrm{i}2|\mu| \int \mathrm{d}^2x[\tilde{b}^+(x)\,\partial_\mu\tilde{b}(x)]\,a_\mu(x). \end{aligned} \tag{16}$$

The function $\tilde{b}^+(x)\,\partial_\mu\tilde{b}(x)$ satisfies

$$\begin{aligned} \tilde{b}^{+1}(x)\partial_\mu\tilde{b}(x) &\equiv r(x)\,\mathrm{e}^{-\mathrm{i}\psi(x)}\,\partial_\mu[r(x)\,\mathrm{e}^{\mathrm{i}\psi(x)}] = \mathrm{i}\tilde{b}^+(x)\tilde{b}(x)[\partial_\mu\psi(x)] + r(x)\,\partial_\mu r(x) \\ &= \mathrm{i}r^2(x)V_\mu^{(\mathrm{n})}(x) + \partial_\mu r^2(x) = \mathrm{i}V_\mu^{(\mathrm{n})}(x)r^2(x)[1 - \mathrm{i}\partial_\mu r^2(x)/2r^2(x)]. \end{aligned} \tag{17}$$

Using the fact that $\partial_\mu r^2(x)/r^2(x) < 1$ we obtain for the last term in equation (16):

$$\begin{aligned} \mathrm{i}2|\mu| \int \mathrm{d}^2x[\tilde{b}^+(x)\,\partial_\mu\tilde{b}(x)]\,a_\mu(x) &\simeq \mathrm{i}2|\mu|\overline{r^2(x)} \int \mathrm{d}^2x V_\mu^{(\mathrm{n})}(x)a_\mu(x) \\ &= \mathrm{i}2|\mu|r^2(x) \int \mathrm{d}^2x\,\Delta_\mu\psi(x)a_\mu(x) = \mathrm{i}2|\mu|\overline{r^2(x)} \\ &\quad \times \left(\psi(x)a_\mu(x)|_{-\infty}^{\infty} - \int \mathrm{d}^2x\,\psi(x)\Delta_\mu a_\mu(x)\right) = 0. \end{aligned} \tag{18}$$

Equation (15) was obtained using the vanishing of $\psi(x)\boldsymbol{a}(x)$ on the boundary and $\nabla \cdot \boldsymbol{a} = 0$. Using equations (16) and (18), instead of equation (15) we have

$$H = H_0 + H_v \tag{19}$$

with

$$H_0 = t\Sigma\bar{b}^+(x)\bar{b}(x+\mu) + \text{HC} + \mu_0\Sigma\bar{b}^+(x)\bar{b}(x) + V\Sigma\bar{b}^+(x)\bar{b}^+(x)\bar{b}(x)\bar{b}(x) \tag{20}$$

and

$$H_v = -t\sum_{x,\mu}\bar{b}^+(x)\bar{b}(x)a_\mu^2(x) = -t\int d^2x\, r^2(x)[\Delta_\mu\Phi(r)]^2 \simeq t\overline{r^2(x)}\int d^2x\,\Phi(r)\Delta^2\Phi(r). \tag{21}$$

Using the spin-coherent representation we have

$$\overline{r^2(x)} = \overline{\rho(x)[1-\rho(x)]} \simeq \bar{\rho}(1-\bar{\rho})$$

where $\bar{\rho}$ is the quasi-condensate. In obtaining equation (21) we have used [5] $\bar{\rho} = \rho(K_0) \leqslant \delta$, which is the quasi-condensate at a finite momentum K_0. (In two dimensions we have to use $\rho(x) = \rho(K_0) + \pi(x)$ with $\langle\pi(x)\rangle = 0$ and $K_0 \sim \mu_0^{\frac{1}{2}}$.) We assume that all the Fourier components $K < K_0$ are contributing to the quasi-condensate as $\rho(K_0)$.

Equation (21) can be further simplified if we use equation (11), $\Delta^2\Phi(r) = 2\pi Q(r)$ with the solution

$$\Phi(r) = \sum_{r'} G(r,r')Q(r').$$

The substitution of this solution in equation (21) gives

$$H_v = 2\pi\bar{t}\bar{\rho}\sum_{r,r'} Q(r)G(r,r')Q(r') \tag{22}$$

where $\bar{t} \equiv t(1-\bar{\rho})$ is the effective hopping.

Equation (22) can be further simplified if we remove the infrared divergences. This is obtained by imposing the charge neutrality condition [6]

$$\Sigma Q(r) = 0 \qquad \Sigma q(r) = -(1/2\pi)\Sigma B(r).$$

Instead of equation (22) we have

$$H_v = 2\pi\bar{t}\bar{\rho}\left(\sum_{r\neq r'} Q(r)\tilde{G}(r,r')Q(r') + (\pi/2)\sum_r Q^2(r)\right) \tag{23}$$

where

$$\tilde{G}(r,r') = -\ln(K_0|r-r'|) \tag{24}$$

$$Q(r) = q(r) + (1/2\pi\Theta)\bar{b}^+(x)\bar{b}(x). \tag{25}$$

The solution of equation (23) changes with Θ. For $\Theta = 1/2\pi$ the bosons become fermions [1]. This can be seen also from the charge equation. For $\Theta = 1/2\pi$ the charge $Q(r)$ is given by $Q(r) = q(r) + \bar{b}^+(x)\bar{b}(x)$.

As a result of the charge neutrality condition we find that binding will occur between the -1 intrinsic vortex and the $+1$ positive vortex induced by the electrical charge $\bar{b}^+(x)\bar{b}(x)$. Here we have the situation where the electric charge that binds with a vortex gives rise to a fermion.

For $\Theta = 1/\pi$ the vorticity is given by $Q(r) = q(r) + \frac{1}{2}\bar{b}^+(x)\bar{b}(x)$. Neutrality of the vortex lattice is obtained if we choose the solutions: $Q(r) = 0, \pm 1/2, \pm 1, \ldots$ ($+1/2$ corresponds to $q(r) = 0$, $\bar{b}^+(x)\bar{b}(x) = 1$; $-1/2$ corresponds to $q(r) = -1$, $\bar{b}^+(x)\bar{b}(x) = 1$; ± 1 corresponds to $q(r) = \pm 1$, $\bar{b}^+(x)\bar{b}(x) = 0$). For this case we can write $Q(r) = \pm\frac{1}{2}\rho(x) + m(r)$, $\rho(x) = \bar{b}^+(x)\bar{b}(x)$ and $m(r)$ is an integral vorticity that satisfies

$\Sigma m(r) = 0$. The fractional vortices $\pm 1/2$ can be expressed in terms of the electric charge $\pm\frac{1}{2}\rho(x) = \frac{1}{2}\rho_\alpha(x)$, $\alpha = +, -$, $\rho_+(x) = \rho(x)$, $\rho_-(x) = -\rho(x)$. At low temperature the $+\frac{1}{2}\rho(x)$ and $-\frac{1}{2}\rho(x)$ vortices are confined into dipoles. Therefore, two electric charges $\rho(x)$ with opposite vortices bind forming a boson of charge $2\rho(x) = 2e$.

Since the neutrality condition $\Sigma m(r) = 0$ is satisfied, the effect of the fluctuating vorticity $m(r)$ is evaluated by integration of these fluctuations [7]. The effect of these fluctuations is to renormalize the vortex potential $\tilde{G}(r, r') \to \tilde{G}_{\text{eff}}(r, r')$. At low temperature this renormalization is not important. Physically one can understand this result in the following way. The $\frac{1}{2}\rho_+(x)$, $-\frac{1}{2}\rho_-(x)$ are confined into dipoles and the effect of the integral vorticity $m(r)$ is to form dipoles with the fractional charges. Since the formation of dipoles $m(r)$ with $\frac{1}{2}\rho_+(r)$ is not possible (fractional charges cannot form dipoles with integral charges [7] we conclude that $\tilde{G}_{\text{eff}}(r, r') \sim \tilde{G}(r, r')$.

At high temperature a phase transition to a plasma phase is characterized by free vortices and therefore free electric charges [6]. The critical temperature for formation of pairs is identical to the Kosterlitz–Thouless transition for $\pm 1/2$ vortices with $T_K \sim 2\pi\bar{\rho}\bar{t} \sim t\delta(1-\delta)$.

In order to study the question of superconductivity one has to consider in addition to binding of vortices the kinetic part given in equation (20).

We use the notation $\rho_\alpha(x) = \tilde{b}^+_\alpha(x)\tilde{b}_\alpha(x)$ with the index $\alpha = +, -$ representing the $\pm$ vorticity of the electric charge. Each electric charge has $+1/2$ or $-1/2$ vorticity. We find an effective model that is similar to previous models, which might have a superconducting phase

$$H = t\sum_{x,\mu,\alpha} \tilde{b}^+_\alpha(x)\tilde{b}_\alpha(x+\mu) + \text{HC} + (\mu_0 + \pi^2\bar{\rho}\bar{t}/4)\sum_{x,\alpha}\tilde{b}^+_\alpha(x)\tilde{b}_\alpha(x)$$
$$- (\pi\bar{\rho}\bar{t}/2)\sum_{x,x',\alpha}\tilde{b}^+_\alpha(x)\tilde{b}^+_{-\alpha}(x')V_{\text{eff}}(x,x')\tilde{b}_{-\alpha}(x')\tilde{b}_\alpha(x). \tag{26}$$

The Hamiltonian given in equation (26) describes a hard-core boson with attractive interaction ($V_{\text{eff}}(x,x') = \tilde{G}_{\text{eff}}(r,r') - V\delta(x-x')$) for bosons with opposite spin α (vorticity). The presence of the term $V\delta(x-x')$ ($V \to \infty$) prevents double occupancy (the hard-core boson condition for bosons with *opposite* vorticity). In addition we have the condition that $(\tilde{b}_\alpha)^2 = (\tilde{b}^+_\alpha)^2 = 0$, the hard-core boson condition for the same vorticity. This condition can be fulfilled if we replace the boson coherent states by Grassmanian coherent states: C_α, C^+_α (fermions). The mapping [1] is performed with the aid of a fictitious vector potential: $A^F_\mu(x) = \chi(x+\mu) - \chi(x)$, $\nabla \times A^F = 2\pi C^+_\alpha(x)C_\alpha(x)$. The relation between the two coherent-states representations is

$$\tilde{b}^+_\alpha(x) = C^+_\alpha(x)\,e^{i\chi(x)} \qquad \tilde{b}_\alpha(x) = e^{-i\chi(x)}C_\alpha(x).$$

The first condition of hard-core boson with opposite vorticity is enforced by the projector

$$P_d = \prod_x [1 - n_+(x)n_-(x)] \qquad n_\pm = C^+_\pm C_\pm.$$

As a result of these transformations we obtain instead of equation (26) the usual model of fermions in a magnetic field with an attractive interaction $V_{\text{eff}}(x,x') = \tilde{G}_{\text{eff}}(r,r')$ (instead of $V_{\text{eff}}(x,x') = \tilde{G}_{\text{eff}}(r,r') - V\delta(x-x')$):

$$H = P_d\tilde{H}P_d$$

$$\bar{H} = t \sum_{x,\mu,\alpha=\pm} C^+_\alpha(x)\,\mathrm{e}^{\mathrm{i}A^{\mathrm{F}}_\mu(x)} C_\alpha(x+\mu) + \mathrm{HC} + (\mu_0 + \pi^2\bar{\rho}\bar{t}/4)\sum_{x,\alpha} C^+_\alpha(x)C_\alpha(x)$$

$$-(\pi\bar{\rho}\bar{t}/2)\sum_{x\neq x'} C^+_+(x)C^+_-(x')\bar{G}_{\mathrm{eff}}(x,x')C_-(x')C_+(x).$$

If we replace A^{F} by the mean-field condition,

$$\times\,\nabla = A^{\mathrm{F}} \simeq 2\pi\langle C^+_\alpha(x)C_\alpha(x)\rangle$$

and

$$\langle C^+_+(x)C_+(x)\rangle = \langle C^+_-(x)C_-(x)\rangle = \tfrac{1}{2}\langle\rho(x)\rangle,$$

we diagonalize the kinetic part. We obtain a complete orthonormal set of Grassmanian numbers, $\bar{C}^+_{l,\alpha}$, $\bar{C}_{l,\alpha}$ obtained from the original basis $C^+_\alpha(x)$, $C_\alpha(x)$ by a unitary transformation, $\bar{C}^+_{l,\alpha} = \Sigma_x\, \psi_l(x)C^+_\alpha(x)$, where $\{\psi_l(x)\}$ are eigenstates of the tight-binding Hamiltonian in a constant magnetic field. Owing to the attractive interaction, a singlet wavefunction for the superconducting state is suggested:

$$|\varphi_{\mathrm{sc}}\rangle = P_{\mathrm{d}}\left(\prod_l \bar{C}^+_{l,+}\bar{C}^+_{l,-}\right)^{N/2}|0\rangle = P_{\mathrm{d}}\left(\sum_{x,x'}\prod_l \psi_l(x)\psi_l(x')C^+_+(x)C^+_-(x')\right)^{N/2}|0\rangle$$

$$= P_{\mathrm{d}}\left(\sum_{x,x'} a(x,x')C^+_+(x)C^+_-(x')\right)^{N/2}|0\rangle.$$

The form of $|\varphi_{\mathrm{sc}}\rangle$ with $a(x,x') \equiv \Pi_l\, \psi_l(x)\psi_l(x')$ is identical to the generalized RVB state introduced by Anderson [8] for the new superconductors.

To conclude, we have shown that a hard-core boson with a statistical fictitious vector potential $A(x)$, which describes the semion state $\Theta = 1/\pi$, leads to the following picture. The presence of the intrinsic vortices in the semion state plus the charge neutrality condition give a binding with $\pm 1/2$ vorticity. Below the Kosterlitz–Thouless transition $T < T_{\mathrm{K}}$ we find an effective model, which might have anisotropic superconductivity.

References

[1] Fradkin E 1989 *Phys. Rev. Lett.* **63** 322
[2] Chen Yi-Hong, Wilczek F, Witten E and Halperin B I 1989 *Preprint*
[3] Negele J W and Orland H 1988 *Quantum Many Particle Systems* (Reading, MA: Addison Wesley)
[4] Balakrishnan R, Sridhar R and Vasudera R 1989 *Phys. Rev.* B **39** 174
[5] Popov V N 1983 Functional Integrals in Quantum Field Theory and Statistical Physics (Dordrecht: Reidel)
[6] Kogut J B 1979 *Rev. Mod. Phys.* **51** (4) 700
[7] Samuel S 1978 *Phys. Rev.* D **18** 1916
[8] Anderson P W 1987 *Science* **235** 1196